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THE LOGARITHMICALLY NORMAL LAW OF DISTRIBUTION OF DIMENSIONS OF PARTICLES WHEN BROKEN INTO SMALL PARTS

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THE LOGARITHMICALLY NORMAL LAW OF DISTRIBUTION OF DIMENSIONS OF PARTICLES WHEN BROKEN INTO SMALL PARTS

A. N. Kolmogoroff

ABSTRACT. The purpose of the present note is to furnish a general diagram of the chance process of gradual fragmentation for which the Gauss law of distribution can be theoretically derived for lagarithms of the dimensions of particles in the limiting value (with unlimited refinement). It would be of interest to investigate the mathematical schemes according to which the rate of fragmentation of particles decreases (or increases) with reduction in dimension.

In a note of N. Rasumovski [1], which recently appeared, many cases were /99* mentioned in which the logarithms of the dimensions of particles (grains of gold in gold sand, lumps of ore when fragmented, etc.) come satisfactorily close to the Gauss law of distribution. The purpose of the present note is to furnish a general diagram of the chance process of gradual fragmentation for which the Gauss law of distribution can be theoretically derived for logarithms of the dimensions of particles in the limiting value (with unlimited refinement). Possibly similar considerations can also lead to a clarification of the applicability of the Gauss distribution to logarithms of ore content in individual tests (the note of N. Rasumovski mentioned above is essentially focussed on this question).

We consider the total amount of particles $N(t)$ and their distribution by dimension in successive moments $t = 0, 1, 2, 3, \ldots$

Let $N(r, t)$ designate the amount of particles with dimensions $\rho \leq r$ at time t (for the following what ρ means is unimportant: the diameter, the gravity or any other property of the dimension of the particle, as long as the condition is met that the dimension of each particle originating as a result of the fragmentation of a particle of dimension r does not exceed r). With $Q(k)$ we designate the mathematical expectation of the number of particles of dimensions $\rho \leq kr$, which arise in the time between t and $t + 1$ from one

*Numbers in the margin indicate pagination in the foreign text.

particle, which had the dimension r at time t, and we posit

$$A=\frac{1}{Q(1)}\int_0^1 \log k\, dQ(k). \tag{1}$$

$$B^2=\frac{1}{Q(1)}\int_0^1 (\log k-A)^2\, dQ(k). \tag{2}$$

Among other assumptions enumerated below, it can be shown that the ratio

$$\frac{N(e^x, t)}{N(t)} \tag{3}$$

for a sufficiently large t with any one probability close to one is differentiated to an arbitrarily slight degree by

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$$\frac{1}{\sqrt{2\pi t}B}\int_{-\infty}^{x} e^{-\frac{(\xi-At)^2}{2B^2t}}\, d\xi \tag{4}$$

The essential assumption which we make as an argument for this relationship consists in the independence of the probability that within the unit of time each particle will be fragmented into one or another number of parts, the dimensions of which stand in one or another relationship to the absolute dimension of the particle. /100

In order to be able to formulate our assumptions more exactly, we shall introduce a few designations. Let p_n be the probability of obtaining exactly n particles in the time between t and t + 1 when a particle is fragmented, and let

$$F_n(a_1, a_2, \ldots, a_n)=P(k_1\leqslant a_1,\ k_2\leqslant a_2,\ \ldots,\ k_n\leqslant a_n)$$

be the relative law of distribution of the ratios $k_i = r_i/1$ of the dimensions of the n particles obtained to the dimension of the original particle. We shall take up the enumeration of the end particles which occur during fragmentation according to the increasing dimensions $r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_n$. Correspondingly the function $F_n(a_1, a_2, \ldots, a_n)$ is defined only for

$$0\leqslant a_1\leqslant a_2\leqslant\ldots\leqslant a_n\leqslant 1$$

we evidently have

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$$Q(k)=\sum_{n=1}^{\infty} P_n \{F_n(k,1,\ldots,1,1)+F_n(k,k,1,\ldots,1,1)+\ldots+ \\ +F_n(k,k,k,\ldots,k,1)+F_n(k,k,\ldots,k,k)\}.$$

We assume that

a) the probabilities P_n and the distributions F_n of the absolute dimensions of the particle do not depend on its previous history (i.e., the previous fragmentation which lead to this particle) nor on the fate of other particles;

b) the mathematical expectation Q(1) of the total number of particles which arise in the time between t and t + 1 from one particle is finite and larger than one;

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c) the integral

$$\int_0^1 |\log k|^3 dQ(k) \qquad (5)$$

is finite;

d) in the initial moment t = 0 a definite number of particles N(0) is available with any distribution according to dimensions N(r, 0).

Among our assumptions are valid:

1) the mathematical expectation of the total number N(t) of the particles at time t is equal to

$$\bar{N}(t)=N(0)Q^t(1); \qquad (6)$$

2) equation (4) for a sufficiently large t with any probability close to one differs to an arbitrarily small degree from the equation

$$\frac{\bar{N}(e^x,t)}{\bar{N}(t)}=\frac{\bar{N}(e^x,t)}{N(0)Q^t(1)}=T(x,t) \qquad (7)$$

of the corresponding mathematical expectation[1].

[1] This circumstance should not be taken as completely evident. For the magnitudes N(t) and N(r,t), taken singly, the analogous assertation of its asymptotic approximation to expectations $\bar{N}(t)$ and $\bar{N}(r,t)$ is false.

Consequently our task consists in the evaluation of T(x,t). From assumptions a) and b) it follows that /101

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$$\overline{N}(r, t+1)=\int_0^1 \overline{N}\left(\frac{r}{k}, t\right) dQ(k). \tag{8}$$

if we posit

$$Q(k)=Q(1)\,S(\log k), \tag{9}$$

we get

$$T(x, t+1)=\int_{-\infty}^{0} T(x-\xi, t)\, dS(\xi). \tag{10}$$

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from (7) and (8).

It easily follows from (7) and (9) that the function S(x) and

$$T(x, 0)=\frac{N(e^x, 0)}{N(0)}$$

(4")

satisfy all conditions which must be demanded of distribution functions[1]. As a result of the recurrent relationship (10), the same is also true in reference to the functions T(x,t) for any complete $t > 0$[2]

It follows from condition c) that the integral

$$\int_{-\infty}^{0} |x|^2\, dS(x) \tag{11}$$

is finite. As a result of (10) and (11), for $t \to \infty$ we have according to Liapounoff's set

$$T(x, t) \to \frac{1}{\sqrt{2\pi t}\, B} \int_{-\infty}^{x} e^{-\frac{(\xi-At)^2}{2B^2 t}} d\xi \tag{12}$$

[1] Here S(x) - S(0) = 1 for x > 0. Therefore in all of the integrals with dS the upper limit 0 can be replaced by ∞ without changing the integral value.

[2] In our problem S(x) and T(x,t) are not probability distributions, but denote definite mathematical expectations. However, this does not interfere with the use of Liapounoff's set, considered as one set of the analysis.

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uniformly in reference to x, where $A = \int_{-\infty}^{0} x dS(x), \quad B^2 = \int_{-\infty}^{0} (x-A)^2 dS(x)$

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obviously coincide with the magnitudes defined by formulae (1) and (2).

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It would be of interest to investigate the mathematical schemes according to which the rate of fragmentation of particles decreases (or increases) with reduction in dimension. It is natural that here the first cases to be considered are those in which the rate of fragmentation is proportional to a power of the dimensions of the particles. If this power is different from zero, it is to be assumed that the logarithmically normal law is no longer applicable.

REFERENCES

1. Rasumovski, N.: *C. R. Acad. Sci. URSS*, Vol. XXVIII, p. 815, 1940.

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